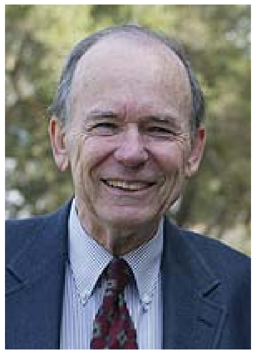
FIPSE ΦΨ-1

Beyond the Luenberger Observer:
New routes for state estimation

Frank Allgöwer

Institute for Systems Theory and Automatic Control
University of Stuttgart



David Luenberger

August 29, 2012

What is State Estimation?

What is State Estimation?

State Estimation determines the underlying behavior of the system at any point in time by using measurable (past and current) information of the system.

History of State Estimation

1638 G. Galilei ("Dialogues concerning two new sciences"): Determine the length of a pendulum by comparing the number of swings with a pendulum of known length

1960 R. Kalman: Introduction of the Kalman filter

1964 D. Luenberger: State estimation of linear systems

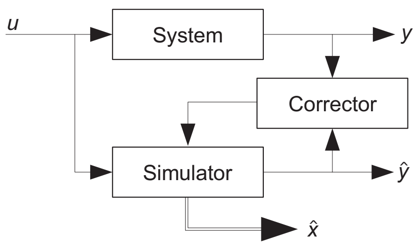
Two main classes of observers:

Deterministic Observers	Stochastic Observers
Disturbances or uncertainties are assumed to be deterministic. Typically: no focus on disturbance behavior	Disturbances and uncertainties are modelled as noise. Typically: Noise suppression as focus.

Focus on deterministic observers in the following!

Standard Luenberger-type Observer

The standard observer structure consists of a simulator and a corrector:



Goal

Reconstruct system state x by using the output y and the simulated output $\hat{y}$

$\Rightarrow$ Estimated state $\hat{x}$

State of the Art

Most observer methods are based on the standard Luenberger observer structure:

- Luenberger observers, e.g. *[Luenberger '64], [Andrieu & Praly '09]*
- Extended Luenberger observers, e.g. *[Birk & Zeitz '97], [Ergueta, et al. '08]*
- High-gain observers, e.g. *[Gauthier & Kupka '01], [Ahrens & Khalil '09], [Bullinger & Allgöwer '97]*
- Sliding-mode observers, e.g. *[Utkin, et al. '99], [Fridman, et al. '11]*
- ...

Mature area with sophisticated approaches for many classes of systems

But: There are many open questions, problems and limitations ...

Observability

Question:

Given a model of a dynamical system and a piece of an output trajectory produced by this system. Can we uniquely determine the system's internal state based upon this information?

$\rightsquigarrow$ leads to the formal concept of observability

Definition: Observability

Consider a system of the form

$$\dot{x} = f(x), \; y = h(x), \; x \in \mathbb{R}^n, \; y \in \mathbb{R}^m$$

with initial condition x_0 at $t = 0$. The system is observable if from $y(t, x_0) = y(t, \tilde{x}_0), \; t > 0$ follows that $x_0 = \tilde{x}_0$.

$\rightsquigarrow$ Answer to question is *YES* if the system is observable.

Checking for Observability: Linear Systems

Nonlinear Case: Checking for observability is a hard task

Theorem (Nonlinear observability)

Consider a nonlinear dynamical system of the form

$$\dot{x} = f(x), \ y = h(x), \ x \in \mathbb{R}^n, \ y \in \mathbb{R}^m$$

wherein $f(\cdot) \in C^{n-1}(\mathbb{R}^n)$ *and* $h(\cdot) \in C^n(\mathbb{R}^n)$. *Let*

$$H(x) = \begin{bmatrix} h \\ L_f(h) \\ \vdots \\ L_f^{n-1}(h) \end{bmatrix}.$$

The system is observable if the mapping $H(\cdot)$ *is univalent in* $\mathbb{R}^n$, *i.e., one-to-one from* $\mathbb{R}^n$ *onto* $H(\mathbb{R}^n)$.

Conclusion on Observability

- Checking a linear system for observability is easy.
- Checking a nonlinear system for observability is a very difficult task, that cannot be performed in most practical applications.

Luenberger-type Observers: State of the Art

- Different system classes can be considered:
  linear, nonlinear, distributed, DAE, . . .
- Complexity is not really an issue
- Advanced system theoretical results available
- Applications include:
  - certainty equivalence based feedback implementation
  - fault detection
  - monitoring
  - identification
  - . . .
- Widely used in industrial practice

Open issues include:

- Limited to asymptotic convergence behavior
- Disturbances not considered in setup
- Separation-principle does not hold for nonlinear systems
- No information about current state estimation-error
- Additional structures needed to handle constraints

Are classical Luenberger-type observers the best choice for addressing the open issues?

Structure of Presentation

- Part 1: Moving Horizon Estimation
- Part 2: Finite Convergence Time Observers
- Part 3: Set-based Estimation

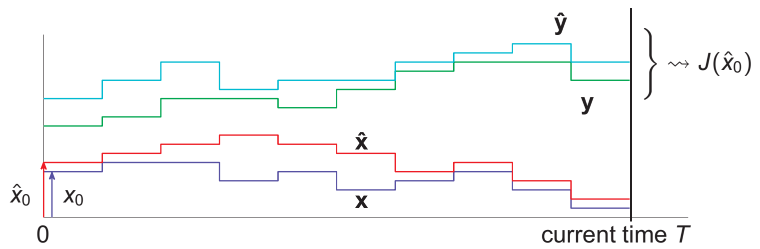
Structure of Presentation

- ➢ **Part 1: Moving Horizon Estimation**
- ➢ **Part 2: Finite Convergence Time Observers**
- ➢ **Part 3: Set-based Estimation**

Optimization Based State Estimation: Basics

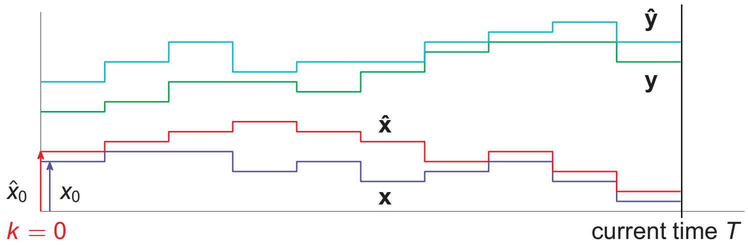
Basic Idea:

1. Given
   - a system model $x^+ = f(x),\ y = h(x)$
   - a measured output trajectory $\mathbf{y} = \{y_k\},\ k \in \{0, 1, \dots, T-1, T\}$
2. For an estimated state initial condition $\hat{x}_0$, use system dynamics to generate corresponding estimated state trajectory $\hat{\mathbf{x}}$ and output trajectory $\hat{\mathbf{y}}$
3. Establish cost function $J(\hat{x}_0)$ to measure how well the estimated output trajectory $\hat{\mathbf{y}}$ fits the measured output trajectory $\mathbf{y}$
4. Optimization of $J(\hat{x}_0)$ over $\hat{x}_0$ yields estimate of initial state x_0



Full Information State Estimation

Full information state estimation: Use all past measurement information starting from $k = 0$.



General properties

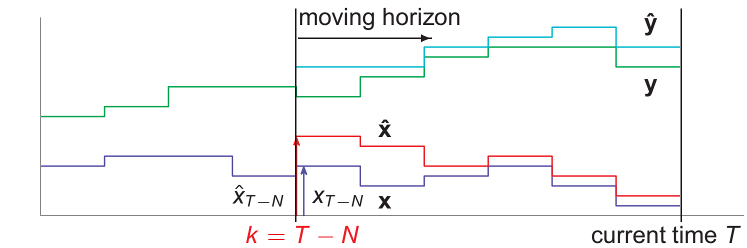
- All measurements are used in optimization.
- Convergence results available for several system setups using different observability conditions
- But: Intractable full information estimator optimization problem!

Hence: Moving horizon estimation (MHE)!

ist

Moving Horizon Estimation (MHE)

Moving horizon estimation: Use information only of N most recent measurements.



General idea

Limit complexity of optimization problem by

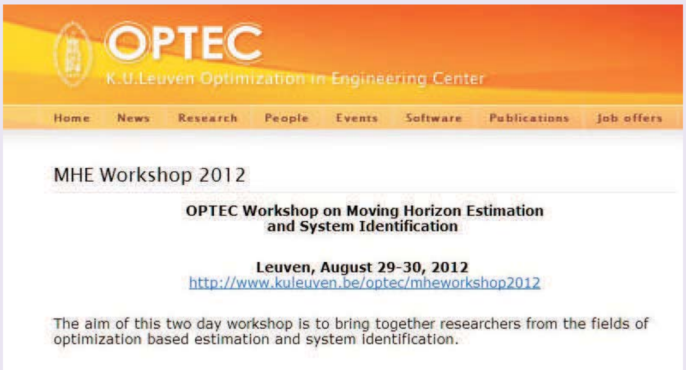
- Restricting time horizon in optimization problem
- Considering information of only N most recent time steps in signals
- Moving this time horizon forward in each time step

- Extensive theory available *[Muske and Rawlings '95], [Michalska and Mayne '96], [Rao, et al. '03], [Farina, et al. '10],....*

ist

Conference on MHE

There is a workshop on Moving Horizon Estimation in parallel to FIPSE-1



A Typical Moving Horizon State Estimation Scheme (1/3)

System typically considered

$$x^+ = f(x,u,w), \; y = h(x,u) + v, \qquad (1)$$

$x \in \mathbb{R}^n$ unknown state, $y \in \mathbb{R}^m$ measured output, $u \in \mathbb{R}^p$ controlled input, $w \in \mathbb{R}^r$ unknown process disturbance, v unknown measurement disturbance.

For some results additional constraints assumed: $x \in \mathcal{X}, \; w \in \mathcal{W}, \; v \in \mathcal{V}$, where $\mathcal{X}, \mathcal{W}, \mathcal{V}$ are closed nonempty sets and $\mathcal{W}, \mathcal{V}$ contain the origin.

Goal: Define MHE and establish convergence result for the estimation error!

Assumptions needed

- Some form of observability/detectability of the System (1)
- Boundedness and convergence of disturbances w and v

A Typical Moving Horizon State Estimation Scheme (2/3)

Cost function

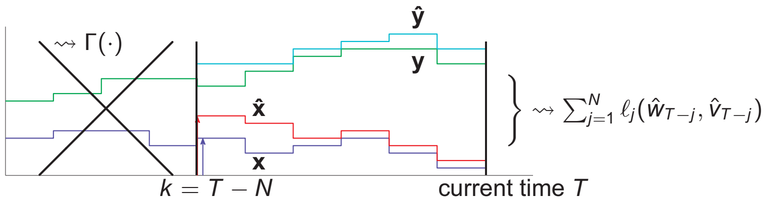
$$J(\hat{x}_{T-N}, \hat{\mathbf{w}}) = \Gamma_{T-N}(\hat{x}_{T-N}) + \sum_{j=1}^{N} \ell_j(\hat{w}_{T-j}, \hat{v}_{T-j}) \quad (2)$$

$$\text{s.t. } \hat{x}^+ = f(\hat{x}, u, \hat{w}), \; y = h(\hat{x}, u) + \hat{v}, \; \text{IC: } \hat{x}_{T-N},$$

wherein $\Gamma(\cdot)$ and $\ell(\cdot)$ satisfy certain typical conditions.

Therein the terms are

- $\Gamma(\cdot)$ the prior weighting function, accounts for discarded part of trajectories
- $\ell(\cdot)$ the stage cost, accounts for fitting error along trajectories



A Typical Moving Horizon State Estimation Scheme (3/3)

A typical convergence result

Theorem

Consider System (1) and cost function (2). Let sufficient assumptions on

- *the System (1),*
- *on the disturbances w and v acting on the system and*
- *on the cost function parameters $\Gamma(\cdot)$ and $\ell(\cdot)$*

be fulfilled. Then repeated optimization of cost function (2) results in a robustly globally asymptotically stable estimator.

Discussion of MHE (1/3)

Advantages of MHE

- Clear goals for estimator can be formulated via cost functional
- Goals can be pursued directly via optimization
- Known (physical) constraints on system states can be employed to improve estimation
- Optimization based framework provides flexibility, e.g. to allow combination with observer of different type

Discussion of MHE (2/3)

Drawbacks of MHE

- Choice of parameters for MHE is elaborate
  - Size of moving horizon N
  - Prior weighting function $\Gamma(\cdot)$
  - Stage costs $\ell(\cdot)$
- Observability/detectability condition is generally hard to check
- Online optimization problem in MHE is numerically expensive, nonconvex, subject to local minima

Discussion of MHE (3/3)

Open issues in MHE

- How can MHE be combined with other observer schemes?
- Extend results towards new system classes
- Which results can be obtained in case of only bounded (but non-convergent) disturbances?
- How can the online optimization be simplified?
  E.g. establish convergence for suboptimal MHE

Summary

Beyond the Luenberger structure...

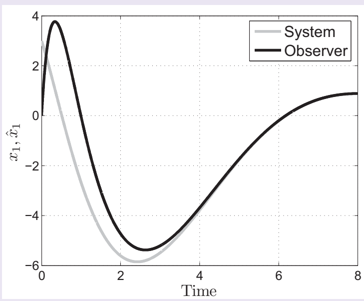
Obviously it is possible to estimate the state of systems without the classical simulator/corrector setup.

Structure of Presentation

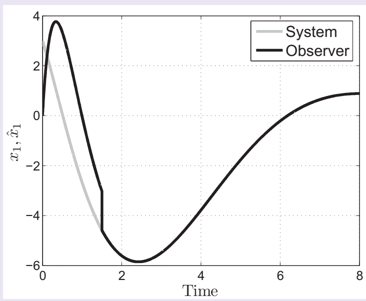
- Part 1: Moving Horizon Estimation
- Part 2: Finite Convergence Time Observers
- Part 3: Set-based Estimation

Motivation

Motivation example



Observer with
asymptotic convergence rate



Observer with
finite convergence time

An observer with finite convergence time is advantageous in some applications:
- Improvement of control performance
- Fast supervision of systems
- . . .

Outline Part 2

1. Luenberger Observer and Observers with Finite Convergence Time
2. An Impulsive Observer with Finite Convergence Time for Linear Systems
3. Extension: Impulsive Observers for Nonlinear Systems
4. Summary and Discussion

Outline Part 2

1. Luenberger Observer and Observers with Finite Convergence Time
    (a) Luenberger Observer
    (b) A Review of Existing Observers with Finite Convergence Time
2. An Impulsive Observer with Finite Convergence Time for Linear Systems
3. Extension: Impulsive Observers for Nonlinear Systems
4. Summary and Discussion

Luenberger Observer

Observer problem

Estimate the state x of the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; y(t) = Cx(t)$$

with $x \in \mathbb{R}^n, u \in \mathbb{R}^p, y \in \mathbb{R}^q$, and the unknown initial condition $x(t_0) = x_0$.

Luenberger observer

The **Luenberger observer** *[Luenberger '66]*

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t))$$

with $\hat{x} \in \mathbb{R}^n$, observer matrix $L \in \mathbb{R}^{n \times q}$, and initial condition $\hat{x}(t_0) = \hat{x}_0$ estimates the state x for arbitrary initial conditions x_0 asymptotically, if the eigenvalues of the matrix $A - LC$ have negative real parts.

The **estimation error** $e = x - \hat{x}$ with $e_0 = x_0 - \hat{x}_0$ has the following properties:

- $\dot{e}(t) = (A - LC)e(t)$
- $e(t) = \exp((A - LC)t)e_0$
- $\lim_{t \to \infty} ||e(t)|| = 0$ (asymptotic (exponential) convergence rate)

ist

An Overview on Observers with Finite Convergence Time (FCT) (1/4)

Observers with FCT

An observer estimates the state of a dynamical system **in finite time** if

$$e(t) = 0 \quad \forall\, t \geq \delta,$$

where $\delta > 0$ is the **convergence time**.

Existing approaches:

- Time-delay techniques *[Engel and Kreisselmeier '02, ...]*
- Filters with finite impulse response structure *[Kwon, et al. '01]*
- The observability matrix (Gramian) *[Medvedev and Toivonen '94], [Byrski '03], ...*
- Sliding mode techniques *[Haskara, et al. '98], [Perruquetti, et al. '07], ...*
- ...

ist

Observers with FCT: Observability Gramian Approach (2/4)

Idea

Suppose that $u = 0$. The output of the linear system is

$$y(\tau) = C\exp(A(\tau - t))x(t).$$

Multiplying both sides by $\exp(A(\tau - t))^T C^T$ and integration from $t - \delta$ to t yields

$$\int_{t-\delta}^{t} (\exp(A(\tau - t))^T C^T y(\tau)d\tau = \left[\int_{t-\delta}^{t} (\exp(A(\tau - t))^T C^T C\exp(A(\tau - t))d\tau\right] x(t).$$

Since the observability gramian is invertible, the state x is estimated in finite time.

Remarks: The implementation requires

- the storage of the output $y(t)$ over the time horizon $[t - \delta, t]$
- the solution of $\int_{t-\delta}^{t} (\exp(A(\tau - t))^T C^T y(\tau)d\tau$ at each time instant t

Observers with FCT: Sliding Mode Approach (3/4)

Idea

Consider, for the sake of simplicity, the linear system

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = u(t), \quad y(t) = x_1(t).$$

The sliding mode observer and its corresponding estimation error dynamics are given by:

$$\begin{aligned} \dot{\hat{x}}_1(t) &= x_2(t) + L_1 \mathrm{sgn}(y(t) - \hat{x}_1(t)) & \dot{e}_1(t) &= e_2(t) - L_1 \mathrm{sgn}(e_1(t)) \\ \dot{\hat{x}}_2(t) &= u(t) + L_2 \mathrm{sgn}(L_1 \mathrm{sgn}(y(t) - \hat{x}_1(t))) & \dot{e}_2(t) &= -L_2 \mathrm{sgn}(L_1 \mathrm{sgn}(e_1(t))) \end{aligned}$$

Using sufficiently high gains L_1, L_2, the observer estimates the state x in finite time.

Remarks:

- The convergence time δ depends on the estimation error e_0
- In practice, there are implementation problems due to chattering

Summary - Observers with FCT (4/4)

- It is possible to design observers with finite convergence time.
- There exist even several approaches.
- All of these approaches require a structure that is different from the classical Luenberger observer.

- High computational complexity for observers that are based on time-delay techniques, the observability Gramian, etc. because of the storage of trajectory pieces and/or the online solution of convolution integrals
- The convergence time or the observer gains depend on the estimation error e_0 for observers that are based on sliding mode techniques, etc.
- In practice, there are implementation problems due to chattering

Outline

Tobias Raff

Computation of the System State - Basic Idea (1/5)

- Two Luenberger observers with $L_1 \neq L_2$ and identical initial conditions

$$\dot{\hat{x}}_1(t) = A\hat{x}_1(t) + Bu(t) + L_1(y(t) - C\hat{x}_1(t)),\ \hat{x}(t_0) = \hat{x}_0$$
$$\dot{\hat{x}}_2(t) = A\hat{x}_2(t) + Bu(t) + L_2(y(t) - C\hat{x}_2(t)),\ \hat{x}(t_0) = \hat{x}_0$$

- At time instant t_1 one obtains the following system of equations:

$$\left.\begin{array}{l} e_1(t_1) = x(t_1) - \hat{x}_1(t_1) \\ e_2(t_1) = x(t_1) - \hat{x}_2(t_1) \end{array}\right\} \ 2n \text{ equations},\ 3n \text{ unknowns}$$

- With $e_i(t_1) = \exp(F_i\delta)e_0,\ e_0 = e_i(0),\ F_i = A - L_iC,\ \delta = t_1 - t_0$, one has

$$\left.\begin{array}{l} \exp(F_1\delta)e_0 = x(t_1) - \hat{x}_1(t_1) \\ \exp(F_2\delta)e_0 = x(t_1) - \hat{x}_2(t_1) \end{array}\right\} \ 2n \text{ equations},\ 2n \text{ unknowns}$$

- At time instant t_1 one obtains the exact state $x(t_1) = \tilde{K}[\hat{x}_1(t_1)^T \hat{x}_2(t_1)^T]^T$

  with $\tilde{K} = (I_n - \exp(F_2\delta)\exp(-F_1\delta))^{-1}\,[-\exp(F_2\delta)\exp(-F_1\delta)\ I_n]$

ist

Observer Scheme - Basic Idea (2/5)

Idea

Update of the states of the two Luenberger observers with $x(t_1)$ at time instant t_1.

Scheme for an observer with finite convergence time δ:

1. Estimation phase for $t < t_1$:

$$\dot{\hat{x}}_i(t) = A\hat{x}_i(t) + Bu(t) + L_i(y(t) - C\hat{x}_i(t)),\ \hat{x}_i(t_0) = \hat{x}_0,\ i = 1, 2$$
$$\hat{x}(t) = \hat{x}_1(t)$$

2. State update at time instant t_1:

$$\hat{x}_i(t_1^+) = x(t_1) = \tilde{K}[\hat{x}_1(t_1)^T\ \hat{x}_2(t_1)^T]^T,\ i = 1, 2$$

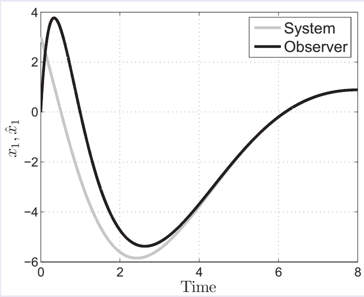
3. Simulation phase for $t > t_1$:

$$\dot{\hat{x}}_i(t) = A\hat{x}_i(t) + Bu(t) + L_i(y(t) - C\hat{x}_i(t)),\ \hat{x}_i(t_1^+) = x(t_1),\ i = 1, 2$$
$$\hat{x}(t) = \hat{x}_1(t)$$

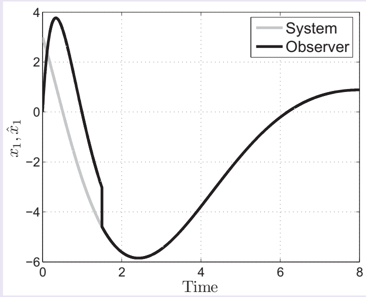
ist

Motivation

Motivation example



Observer with
asymptotic convergence rate



Observer with
finite convergence time

Observer Structure - Basic Idea (3/5)

The proposed observer with finite convergence time is given by

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Gu(t) + Hy(t), && t \neq t_k, \\ z(t_k^+) &= K_k z(t_k), && t = t_k, \\ z(t_0^+) &= z_0, && k = 1, 2, \ldots \\ \hat{x}(t) &= Nz(t), \end{aligned}$$

where $z = [\hat{x}_1^T \ \hat{x}_2^T]^T$ is the observer state, $z_0 = M\hat{x}_0$ the initial condition, and t_k a time sequence satisfying $0 < t_0 < t_1 < \ldots < t_k$ with $\delta = t_{i+1} - t_i$ and $\lim_{k\to\infty} t_k = \infty$.

Remarks:

- $$F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}, G = \begin{bmatrix} B \\ B \end{bmatrix}, H = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, M = \begin{bmatrix} I_n \\ I_n \end{bmatrix}, N = \begin{bmatrix} I_n \\ 0_n \end{bmatrix}^T, \begin{matrix} K_1 = M\tilde{K} \\ K_k = I_{2n}, \end{matrix} \ k = 2, 3, \ldots$$
- The state is updated only at time instant t_1 (updates at $t_2, t_3, \ldots$ also possible)
- The observer exhibits impulsive dynamical behavior (impulsive observer)

Existence - Basic Idea (4/5)

Theorem 1 *[Raff and Allgöwer '07]*

The proposed observer reconstructs the state of the linear system in finite time $\delta > 0$ independent of the initial estimation error e_0, i.e. $e(t) = 0 \ \forall t > t_1, \ e_0 \in \mathbb{R}^n$, if the observer matrices L_1, L_2 and the convergence time δ are chosen such that the eigenvalues of the matrix F have negative real parts and the observer matrix

$$K_1 = [I_n \quad I_n]^T (I_n - \exp(F_2\delta)\exp(-F_1\delta))^{-1} [-\exp(F_2\delta)\exp(-F_1\delta) \quad I_n]$$

exists.

Remarks:

- The observer exists if the linear system is observable and the matrix K_1 exists
- The convergence time does not depend on the estimation error e_0
- The design parameters are the matrices L_1, L_2 ($L_1 \neq L_2$) and δ

Theorem 2 *[Raff and Allgöwer '07]*

For any convergence time δ there exist matrices L_1, L_2 such that the matrix K_1 exists.

Remark:

- In theory, the convergence time δ can be chosen arbitrarily short.

ist

Summary - Basic Idea (5/5)

- Using an impulsive observer (DAE-System) allows to estimate the state of a linear system exactly in arbitrary short time.

What are the further properties of this observer?

ist

Output-Feedback - Analysis (1/2)

- State feedback $u = -R\hat{x}$ in conjunction with the proposed observer
- The dynamics of the closed-loop system (only one update at time instant t_1) is

$$\begin{aligned} \dot{x}(t) &= (A - BR)x(t) + BRe_1(t) \quad t \neq t_1 \\ \dot{e}_1(t) &= (A - L_1C)e_1(t) \\ \dot{e}_2(t) &= (A - L_2C)e_2(t) \\ x(t_1^+) &= x(t_1) \qquad t = t_1 \\ e_1(t_1^+) &= 0, \\ e_2(t_1^+) &= 0. \end{aligned}$$

- The closed-loop system is asymptotically stable if the eigenvalues of $A - BR$, $A - L_1C$, and $A - L_2C$ have negative real parts

A separated design of the state feedback and the proposed observer is possible.

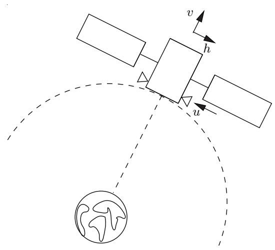
ist

Analysis – Summary (2/2)

- The impulsive observer has three design parameters, namely L_1, L_2, and δ:
  - The parameter δ specifies the convergence time
  - The matrix L_1 is used to tune the transient behavior of e for $t_0 < t < t_1$
  - The matrix L_2 is used to guarantee the existence of the matrix K_1
- The convergence time δ
  - can be chosen arbitrarily short
  - is independent of the estimation error e_0
- Low computational complexity of the proposed observer compared to the computational complexity of other existing observers with finite convergence time
- Similar system theoretical properties as a Luenberger observer
- No exact state estimation in presence of disturbances

ist

Satellite - Example (1/2)



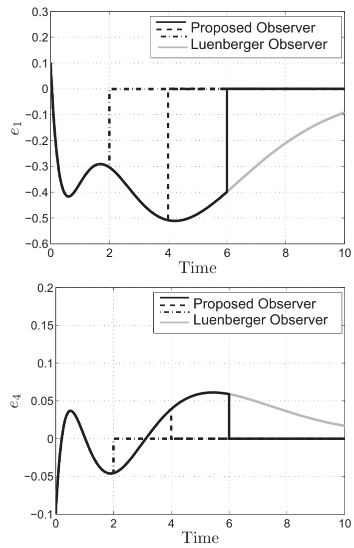
- Station-keeping satellite (weather satellite)

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x$$

- x_1, x_3 represent the coordinates v, h
- Parameter $\omega = \frac{2\pi}{24}$ is the earth's angular velocity
- Horizontal position h is measured
- Eigenvalues of A: $\lambda(A) = [0\ 0\ 0 \pm 0.26j]$
- No control action, i.e. $u = 0$

Simulation Results - Example (2/2)



- Proposed observer:
  - Convergence times: $\delta = 6$ (solid), $\delta = 4$ (dashed), $\delta = 2$ (dash-dotted)
  - Eigenvalues of F_1,F_2: $\lambda(F_1) = [-0.5 - 0.8 - 1.0 - 1.4]$ $\lambda(F_2) = [-2.6 - 2.9 - 3.1 - 3.4]$
- Luenberger observer:
  - Eigenvalues of $F = A - LC$: $\lambda(F) = [-0.5 - 0.8 - 1.0 - 1.4]$

The observer estimates the system state exactly in finite time.

Measurement Disturbances - Analysis II (1/2)

- Output equation is $y(t) = Cx(t) + v(t)$, where $|v(t)| \leq v_{\max}\ \forall t > 0$
- For $t < t_1$ and $t >> t_1$ the observer performs like a Luenberger observer
- At time instant t_1, the estimation error satisfies

$$||e(t_1^+)|| \leq \frac{v_{\max}}{1 - p(\alpha_1\alpha_2)} \left(\frac{p(\alpha_1, \alpha_2)\alpha_1^{n-2}c||L_1||}{-\lambda_1} + \frac{\alpha_2^{n-2}c||L_2||}{-\lambda_1} \right)$$

with $p(\alpha_1, \alpha_2) = (\alpha_1\alpha_2)^{n-1}c^2 \exp((\lambda_1\alpha_2 - \lambda_n\alpha_1)\delta)$

- The observer performs better than a Luenb. observer with obs. matrix L_1 if

$$||e_0|| \geq \exp(-\lambda_1\alpha_1\delta)\frac{v_{\max}}{1 - p(\alpha_1\alpha_2)} \left(\frac{p(\alpha_1, \alpha_2)\alpha_1^{n-2}c||L_1||}{-\lambda_1} + \frac{\alpha_2^{n-2}c||L_2||}{-\lambda_1} \right),$$

i.e. it holds: $||e_L(t)|| \geq ||e(t)||\ \forall t \geq t_0$.

Model Uncertainties - Analysis II (2/2)

- Uncertain system is:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(x(t), u(t), t),$$

where $||w(x(t), u(t), t)|| \leq w_{\max}\ \forall t \geq 0$

- At time instant t_1, the estimation error satisfies

$$||e(t_1^+)|| \leq \frac{w_{\max}}{1 - p(\alpha_1\alpha_2)} \left(\frac{p(\alpha_1, \alpha_2)\alpha_1^{n-2}c}{-\lambda_1} + \frac{\alpha_2^{n-2}c}{-\lambda_1} \right)$$

- Like a Luenberger observer, the observer does not converge asymptotically, i.e., $\lim_{t\to\infty} ||e(t)|| \leq \frac{\alpha_1^{n-2}cw_{\max}}{-\lambda_1}$

No exact state estimation in presence of disturbances.

Outline Part 2

1. Luenberger Observer and Observers with Finite Convergence Time
2. An Impulsive Observer with Finite Convergence Time for Linear Systems
3. Extension: Impulsive Observers for Nonlinear Systems
   - (a) Nonlinear Systems in Observer Normal Form (Example)
   - (b) Nonlinear Systems in Observability Normal Form (Example)
4. Summary and Discussion

Nonlinear Systems in Observer NF (1/3)

Problem

Design an observer with FCT for nonlinear systems in observer normal form (NF)

$$\dot{x}(t) = Ax(t) + \gamma(u(t), y(t)), \quad x(t_0) = x_0$$
$$y(t) = Cx(t),$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ the input, $y \in \mathbb{R}$ the measurable output, $x(t_0) = x_0$ the unknown initial condition, and $\gamma : \mathbb{R}^p \times \mathbb{R}^1 \to \mathbb{R}^n$ is a locally Lipschitz nonlinearity that depends on known arguments. The system matrices are

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Nonlinear Systems in Observer NF (2/3)

An impulsive normal form observer is given by

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Hy(t) + M\gamma(u(t), y(t)), & t &\neq t_k, \\ z(t_k^+) &= K_k z(t_k), & t &= t_k, \\ z(t_0^+) &= z_0, & k &= 1, 2, \dots \\ \hat{x}(t) &= Nz(t). \end{aligned}$$

Theorem 3 *[Raff and Allgöwer '08]*

The nonlinear observer reconstructs the state of nonlinear systems in observer normal form in finite time $\delta > 0$ independent of the initial estimation error e_0, i.e. $e(t) = 0 \ \forall t > t_1, \ e_0 \in \mathbb{R}^n$, if the observer matrices L_1, L_2 and the convergence time δ are chosen such that the eigenvalues of the matrix F have negative real parts and the following observer matrix exists:

$$K_1 = \begin{bmatrix} I_n & I_n \end{bmatrix}^T \left(I_n - \exp(F_2\delta)\exp(-F_1\delta) \right)^{-1} \begin{bmatrix} -\exp(F_2\delta)\exp(-F_1\delta) & I_n \end{bmatrix}.$$

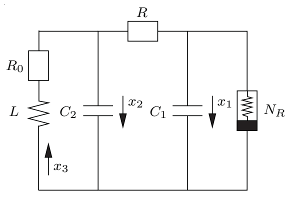
Remarks - Nonlinear Systems in Observer NF (3/3)

- The existence conditions of the impulsive normal form observer are similar to the conditions of the impulsive observer for linear systems
- If the nonlinear system

$$\dot{\bar{x}}(t) = f(\bar{x}(t), u(t)), \quad y(t) = h(\bar{x}(t))$$

  is not given in observer normal form, one can use a state transformation (*[Krener and Isidori '83]*).

Chua's Circuit - Example (1/2)

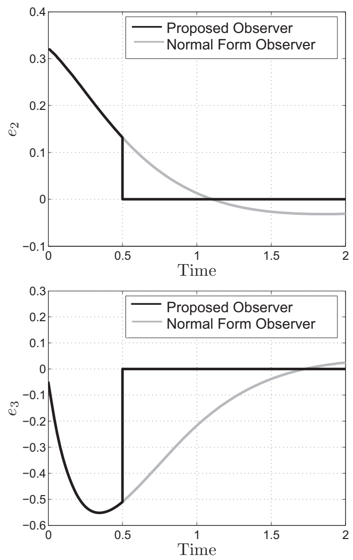


- Chua's circuit (application: cryptosystems)

$$\dot{x}_1 = -a_1x_1 + a_2x_2 + a_3(|x_1 + a_4| - |x_1 - a_4|)$$
$$\dot{x}_2 = x_1 - x_2 + x_3$$
$$\dot{x}_3 = -a_5x_2$$
$$y = x_1$$

- x_1, x_2 and x_3 represent the voltages of the capacitors and the current of the inductor
- Parameter values: $a_1 = 18/7, a_2 = 9, a_3 = -2$, $a_4 = 1$, and $a_5 = 14.286$
- Voltage of the capacitor C_1 is measured
- Chua's circuit can be transformed into observer NF

Simulation Results - Example (2/2)



- Proposed observer:
  - Convergence time: $\delta = 0.5$
  - Eigenvalues of F_1,F_2:
    $\lambda(F_1) = [-1 - 2 - 3]$
    $\lambda(F_2) = [-7 - 8 - 9]$
- Normal form observer:
  - Eigenvalues of $F = A - LC$:
    $\lambda(F) = [-1 - 2 - 3]$

The observer estimates the state of systems in observer normal form in finite time.

Nonlinear Systems in Observability NF (1/4)

Problem

Estimate the state of a nonlinear system in observability NF

$$\dot{x}(t) = Ax(t) + E\phi(x(t), U(t)), \quad x(t_0) = x_0$$
$$y(t) = Cx(t),$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ the input, $y \in \mathbb{R}$ the output, $x(t_0) = x_0$ the unknown initial condition, $U = [u\ \dot{u}\ u^{(2)} \cdots\ u^{(n)}]^T$ is the stacked vector of input derivatives, and $\phi(x, U)$ is a locally Lipschitz nonlinearity. It is assumed that the nonlinearity ϕ is globally bounded, i.e. $|\phi(x, U)| \leq \phi_{\max}\ \forall\ x,\ U$. The system matrices are given by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}^T, \ E = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Nonlinear Systems in Observability NF (2/4)

An impulsive high gain observer (HGO) is given by

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Hy(t), & t &\neq t_k, \\ z(t_k^+) &= K_k z(t_k), & t &= t_k, \\ z(t_0^+) &= z_0, & k &= 1, 2, .. \\ \hat{x}(t) &= Nz(t) \end{aligned}$$

with observer matrices $L_i = \begin{bmatrix} \alpha_i a_{n-1} & \alpha_i^2 a_{n-2} & \cdots & \alpha_i^{n-1} a_1 & \alpha_i^n a_0 \end{bmatrix}^T$.

Remarks:

- $||e(t_1^+)|| \leq \frac{1}{1-p(\alpha_1,\alpha_2)} \left(\frac{c\phi_{\max}}{-\alpha_2\lambda_1}\right) + \frac{p(\alpha_1,\alpha_2)}{1-p(\alpha_1,\alpha_2)} \left(\frac{c\phi_{\max}}{-\alpha_1\lambda_1}\right)$ (can be made small)
- The impulsive HGO
  - does not estimate the system state exactly, i.e. $\lim_{t\to\infty} ||e(t)|| \leq \frac{\phi_{\max} c}{-\alpha_1\lambda_1}$
  - has a better performance than a HGO, i.e. $||e_{\mathrm{HGO}}(t)|| \geq ||e(t)||\ \forall t \geq t_0$
  - converges exponentially if the observer dynamics is replaced by $\dot{z} = Fz + Hy + \Phi(z, U)$ and the nonlinearity is bounded and Lipschitz

Nonlinear Systems in Observability NF (3/4)

An impulsive sliding mode high gain observer (ISMHGO) is given by

$$
\begin{aligned}
\dot{z}_{1_1}(t) &= z_{1_2}(t) - \alpha_1 a_{n-1}(y(t) - z_{1_1}(t)) + L_{s_1} h(t-t_1)\mathrm{sgn}(\xi_1(t))\\
\dot{z}_{1_2}(t) &= z_{1_3}(t) - \alpha_1^2 a_{n-2}(y(t) - z_{1_1}(t)) + L_{s_2} h(t-t_2)\mathrm{sgn}(\xi_2(t))\\
&\dots\\
\dot{z}_{1_n}(t) &= -\alpha_1^n a_0(y(t) - z_{1_1}(t)) + L_{s_n} h(t-t_n)\mathrm{sgn}(\xi_n(t))\\
\dot{z}_{2_1}(t) &= z_{1_2}(t) - \alpha_2 a_{n-1}(y(t) - z_{2_1}(t))\\
\dot{z}_{2_2}(t) &= z_{2_3}(t) - \alpha_2^2 a_{n-2}(y(t) - z_{2_1}(t))\\
&\dots\\
\dot{z}_{2_n}(t) &= -\alpha_2^n a_0(y(t) - z_{2_1}(t))\\
z(t_k^+) &= K_k z(t_k)\\
z(t_0^+) &= z_0\\
\hat{x}(t) &= Nz(t)
\end{aligned}
$$

where $L_{s_i} \in \mathbb{R}$, $\xi_1 = y - z_{1_1}$, and $\xi_i = L_{i-1}\mathrm{sgn}(\xi_{i-1}),\ i = 2, \dots, n$.

Remarks - Nonlin. Systems in Observability NF (4/4)

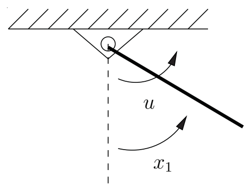
- The observer estimates the system state exactly in finite time $\Delta > \delta$, i.e.

$$e(t) = 0 \quad \forall t > \Delta$$

- The convergence time Δ can be chosen arbitrarily fast (higher gains L_{s_i})
- The convergence time and the gains L_{s_i} do not depend on the estimation error e_0, in contrast to the sliding mode observer *[Haskara, et al. '98]*

The impulsive sliding mode high gain observer combines the advantages of the impulsive HGO (independence of e_0) and of the sliding mode observer (robustness).

Pendulum - Example (1/2)

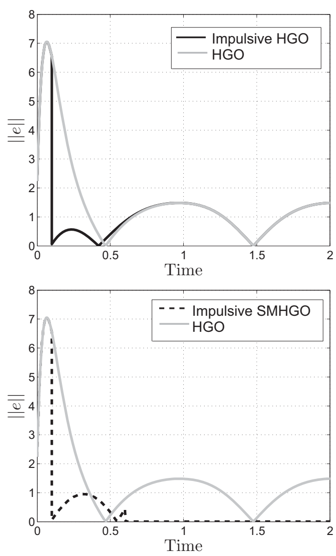


- Pendulum system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) + \frac{1}{ml^2}u$$
$$y = x_1$$

- x_1 represents the angle of the pendulum
- Angle of the pendulum is measured
- Parameter values: $g = 9.81$, $l = 0.9$, and $m = 1.1$
- No control action, i.e. $u = 0$

Simulation Results - Example (2/2)



- Proposed observers:
  - Impulsive HGO:
    - Update time instant: $t_1 = 0.1$
    - Gains: $\alpha_1 = 10, \alpha_2 = 200$
  - Impulsive SMHGO:
    - Convergence time: $\Delta = 1.1$
    - Gains: $\alpha_1 = 10, \alpha_2 = 200$
      $L_{s_1} = 6, L_{s_2} = 91$
- High gain observer:
  - Gain: $\alpha = 10$

The proposed observers outperform high gain observers.

Summary and Discussion

Using new observer structures, it is possible to design observers with predetermined finite convergence time (FCT)

- that are based on
  - two Luenberger observers
  - and one (multiple) state update(s)
- that have similar system theoretical properties as a Luenberger observer with respect to
  - measurement disturbances and model uncertainties
  - output feedback design (separation principle)
- that have a low computational complexity compared to existing observers with FCT
- whose structure can be extended to other system classes (including nonlinear systems)

Structure of Presentation

- Part 1: Moving Horizon Estimation
- Part 2: Finite Convergence Time Observers
- Part 3: Set-based Estimation

Motivation

Problem of Practical Relevance

State estimation in presence of exogenous disturbances

Usual approach:

Use stochastic modeling/filtering techniques.
However:

- not always possible or reasonable
- sometimes give unsatisfactory results
- do not give hard bounds of estimation error

Set-Based Estimation

Alternative Solution

Model uncertainties as unknown but bounded and use **set-valued filter techniques**.

Finds applications, e.g. in

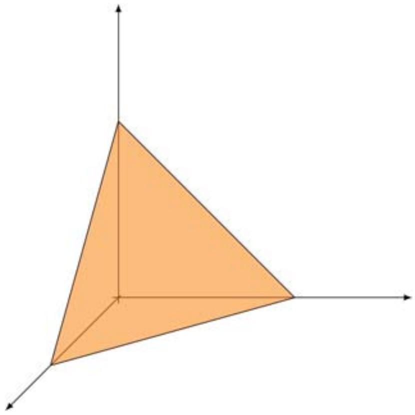
- robust fault detection
- model invalidation
- signal processing
- ...

Idea:

Two-step filter algorithm consisting of

- set-valued prediction
- correction steps

Most Simple Case: LTI System Without Disturbances



System

$$x_{k+1} = Ax_k \qquad y_k = Cx_k \qquad \det(A) \neq 0$$

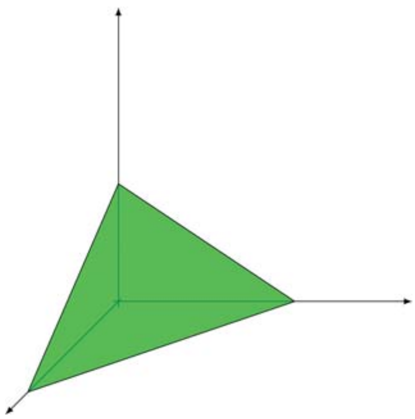
Set-Based Estimation

$$x_1 \in \{x \mid Cx = y_1\}$$

$$x_2 \in \left\{x \;\middle|\; \begin{aligned} CA^{-1}x &= y_1 \\ Cx &= y_2 \end{aligned}\right\}$$

$$x_3 \in \left\{x \;\middle|\; \begin{aligned} CA^{-2}x &= y_1 \\ CA^{-1}x &= y_2 \\ Cx &= y_3 \end{aligned}\right\}$$

Most Simple Case: LTI System Without Disturbances



System

$$x_{k+1} = Ax_k \qquad y_k = Cx_k \qquad \det(A) \neq 0$$

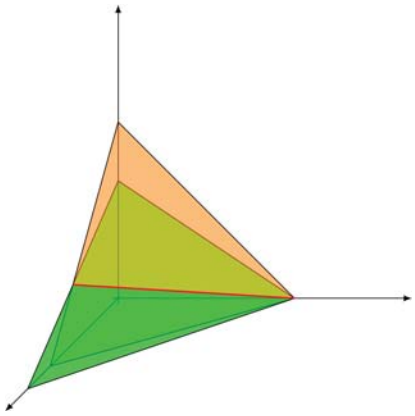
Set-Based Estimation

$$x_1 \in \{x \mid Cx = y_1\}$$

$$x_2 \in \left\{x \;\middle|\; \begin{aligned} CA^{-1}x &= y_1 \\ Cx &= y_2 \end{aligned}\right\}$$

$$x_3 \in \left\{x \;\middle|\; \begin{aligned} CA^{-2}x &= y_1 \\ CA^{-1}x &= y_2 \\ Cx &= y_3 \end{aligned}\right\}$$

Most Simple Case: LTI System Without Disturbances



$$\{CA^{-1}x = y_1\} \cap \{Cx = y_2\}$$

System

$$x_{k+1} = Ax_k \qquad y_k = Cx_k \qquad \det(A) \neq 0$$

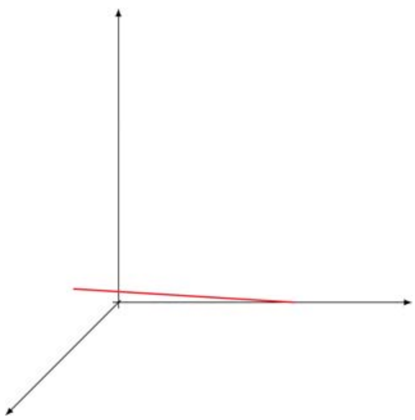
Set-Based Estimation

$$x_1 \in \{x \,|\, Cx = y_1\}$$

$$x_2 \in \left\{ x \,\middle|\, \begin{array}{r} CA^{-1}x = y_1 \\ Cx = y_2 \end{array} \right\}$$

$$x_3 \in \left\{ x \,\middle|\, \begin{array}{r} CA^{-2}x = y_1 \\ CA^{-1}x = y_2 \\ Cx = y_3 \end{array} \right\}$$

Most Simple Case: LTI System Without Disturbances



$$\{CA^{-1}x = y_1\} \cap \{Cx = y_2\}$$

System

$$x_{k+1} = Ax_k \qquad y_k = Cx_k \qquad \det(A) \neq 0$$

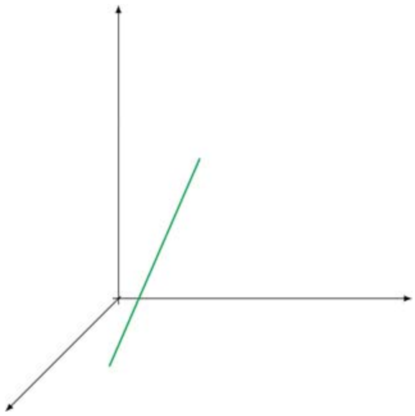
Set-Based Estimation

$$x_1 \in \{x \,|\, Cx = y_1\}$$

$$x_2 \in \left\{ x \,\middle|\, \begin{array}{r} CA^{-1}x = y_1 \\ Cx = y_2 \end{array} \right\}$$

$$x_3 \in \left\{ x \,\middle|\, \begin{array}{r} CA^{-2}x = y_1 \\ CA^{-1}x = y_2 \\ Cx = y_3 \end{array} \right\}$$

Most Simple Case: LTI System Without Disturbances



$A(\{CA^{-1}x = y_1\} \cap \{Cx = y_2\}) =$
$\{CA^{-2}x = y_1\} \cap \{CA^{-1}x = y_2\}$

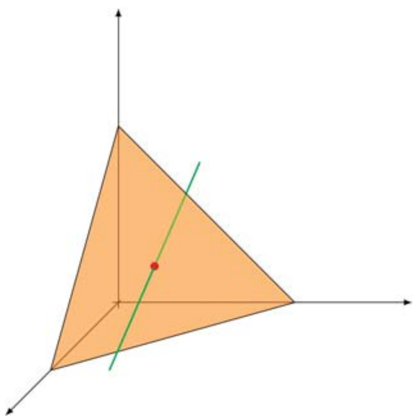
System

$$x_{k+1} = Ax_k \qquad \det(A) \neq 0$$
$$y_k = Cx_k$$

Set-Based Estimation

$$x_1 \in \{x \,|\, Cx = y_1\}$$
$$x_2 \in \left\{ x \,\middle|\, \begin{array}{r} CA^{-1}x = y_1 \\ Cx = y_2 \end{array} \right\}$$
$$x_3 \in \left\{ x \,\middle|\, \begin{array}{r} CA^{-2}x = y_1 \\ CA^{-1}x = y_2 \\ Cx = y_3 \end{array} \right\}$$

Most Simple Case: LTI System Without Disturbances



$\{CA^{-2}x = y_1\} \cap$
$\{CA^{-1}x = y_2\} \cap \{Cx = y_3\}$

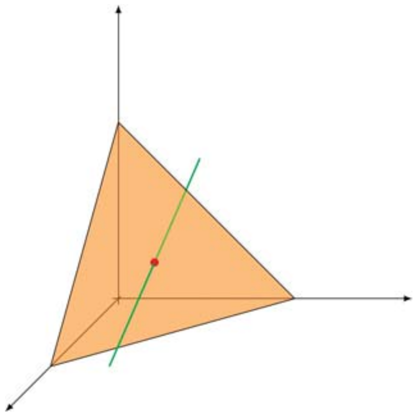
System

$$x_{k+1} = Ax_k \qquad \det(A) \neq 0$$
$$y_k = Cx_k$$

Set-Based Estimation

$$x_1 \in \{x \,|\, Cx = y_1\}$$
$$x_2 \in \left\{ x \,\middle|\, \begin{array}{r} CA^{-1}x = y_1 \\ Cx = y_2 \end{array} \right\}$$
$$x_3 \in \left\{ x \,\middle|\, \begin{array}{r} CA^{-2}x = y_1 \\ CA^{-1}x = y_2 \\ Cx = y_3 \end{array} \right\}$$

Most Simple Case: LTI System Without Disturbances



$\{CA^{-2}x = y_1\} \cap$
$\{CA^{-1}x = y_2\} \cap \{Cx = y_3\}$

System

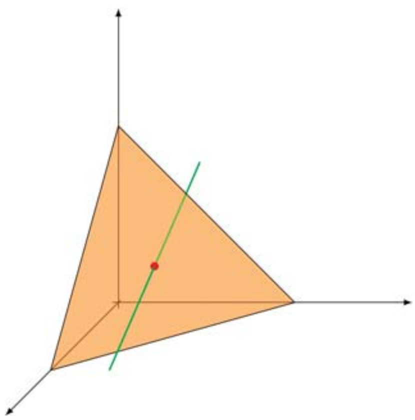
$$x_{k+1} = Ax_k \qquad \det(A) \neq 0$$
$$y_k = Cx_k$$

Set-Based Estimation

$$x_n \in \left\{ x' \;\middle|\; \begin{aligned} CA^{-(n-1)}x &= y_1 \\ &\vdots \\ CA^{-1}x &= y_{n-1} \\ Cx &= y_n \end{aligned} \right\}$$

ist

Most Simple Case: LTI System Without Disturbances



$\{CA^{-2}x = y_1\} \cap$
$\{CA^{-1}x = y_2\} \cap \{Cx = y_3\}$

System

$$x_{k+1} = Ax_k \qquad \det(A) \neq 0$$
$$y_k = Cx_k$$

Set-Based Estimation

$$\underbrace{\begin{bmatrix} CA^{-(n-1)} \\ \vdots \\ CA^{-1} \\ C \end{bmatrix}}_{\text{(Observability Matrix)}\times A^{-(n-1)}} x_n = \begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}$$

ist

Outline Part 3

1. Motivation
2. Linear Systems Without Disturbances
3. General Problem Setup
4. Idea: Set Propagation
5. Computation: Ellipsoidal Sets and Sum of Squares Programming
6. Simulation Example
7. Fault Detection
8. Summary and Discussion

General Problem Setup

Nonlinear, time-varying discrete time system

$$x_{k+1} = f_k(x_k, w_k) \qquad x_0 \in \mathbb{X}_0 \subseteq \mathbb{X}$$
$$y_k = h_k(x_k, v_k)$$

- system state $x \in \mathbb{X} \subseteq \mathbb{R}^{n_x}$
- measured output $y \in \mathbb{R}^{n_y}$
- **bounded** disturbances $w_k \in \mathcal{W}_k \subseteq \mathbb{R}^{n_w}$ and $v \in \mathcal{V}_k \subseteq \mathbb{R}^{n_v}$

Goal

Reconstruct set of possible system states from measurements

General Problem Setup

Nonlinear, time-varying discrete time system

$$x_{k+1} = f_k(x_k, w_k) \qquad x_0 \in \mathbb{X}_0 \subseteq \mathbb{X}$$
$$y_k = h_k(x_k, v_k)$$

- system state $x \in \mathbb{X} \subseteq \mathbb{R}^{n_x}$
- measured output $y \in \mathbb{R}^{n_y}$
- **bounded** disturbances $w_k \in \mathcal{W}_k \subseteq \mathbb{R}^{n_w}$ and $v \in \mathcal{V}_k \subseteq \mathbb{R}^{n_v}$

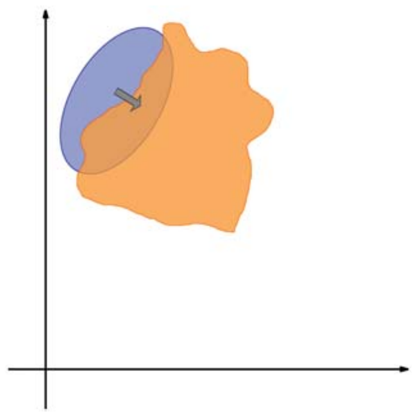
Goal

Reconstruct set of possible system states from measurements

Idea: Set Propagation

System dynamics (prediction)

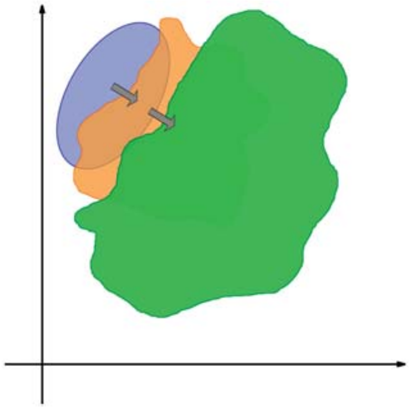
$$x_{k+1} = f_k(x_k, w_k)$$



Idea: Set Propagation

System dynamics (prediction)

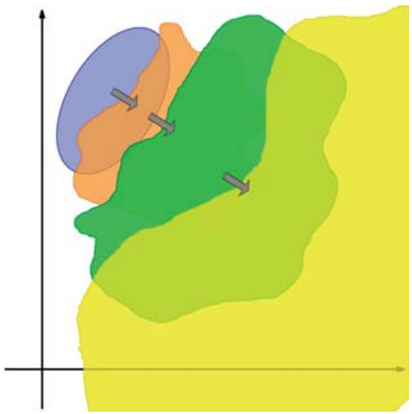
$$x_{k+1} = f_k(x_k, w_k)$$



Idea: Set Propagation

System dynamics (prediction)

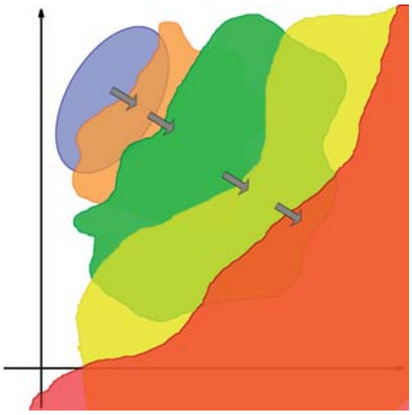
$$x_{k+1} = f_k(x_k, w_k)$$



Idea: Set Propagation

System dynamics (prediction)

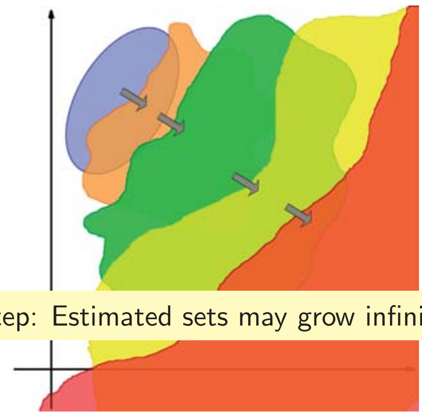
$$x_{k+1} = f_k(x_k, w_k)$$



Idea: Set Propagation

System dynamics (prediction)

$$x_{k+1} = f_k(x_k, w_k)$$

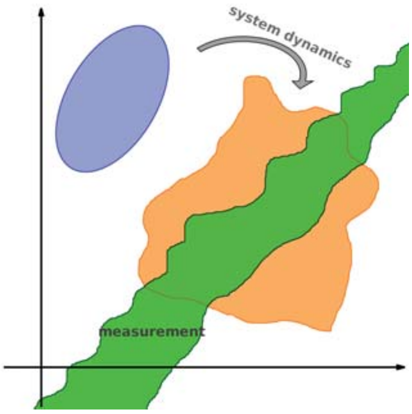


No correction step: Estimated sets may grow infinitely large.

Idea: Set Propagation

Measurement (correction)

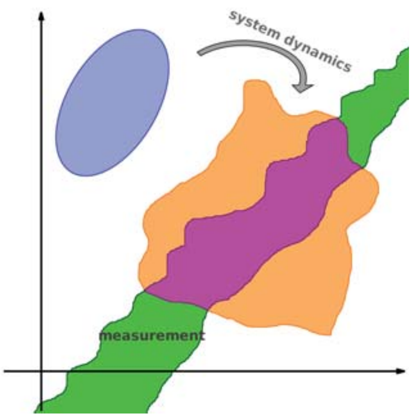
$$y_k = h_k(x_k, v_k)$$



Idea: Set Propagation

Measurement (correction)

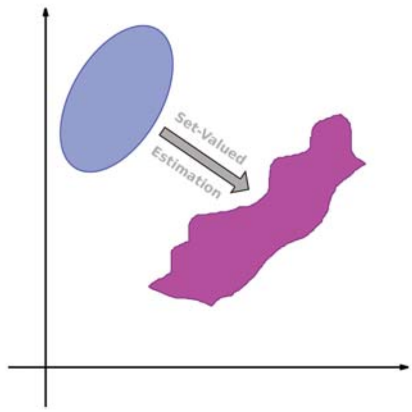
$$y_k = h_k(x_k, v_k)$$



Idea: Set Propagation

Measurement (correction)

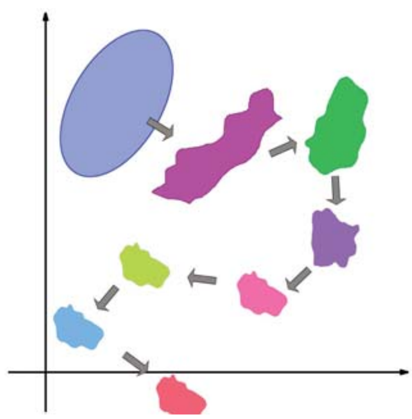
$$y_k = h_k(x_k, v_k)$$



Idea: Set Propagation

System Theoretic Properties

With correction step (and some observability property):



Estimated sets stay bounded.

Set-Based Estimation

- Because of persistent disturbances the state estimates cannot shrink to a point.
- A desirable observer performance consists of the estimated state sets to stay bounded and small.
- No observer *design* needed. Observer computes all possible states that are consistent with the measurements.
- Big challenge is computation of sets.

Computation of Sets

Computation of Sets

- Ideal case: Compute *exact* sets
- Problems:
  - Sets might become intractably complex
  - How to represent sets?
- Solution: Use (typically conservative) outer approximations (e.g. ellipsoids).

Computation of State Sets for Polynomial Nonlinear Systems

- Considered Problem Setup
- SOS-Based Set Computations
  - Approximation by Ellipsoidal Sets
  - Computation of Ellipsoids by SOS-Programming
- Simulation Results

Christoph Maier

Considered Problem Setup: Polynomial Description

Nonlinear, time-varying discrete time system

$$x_{k+1} = f_k(x_k, w_k)$$
$$y_k = h_k(x_k, v_k)$$

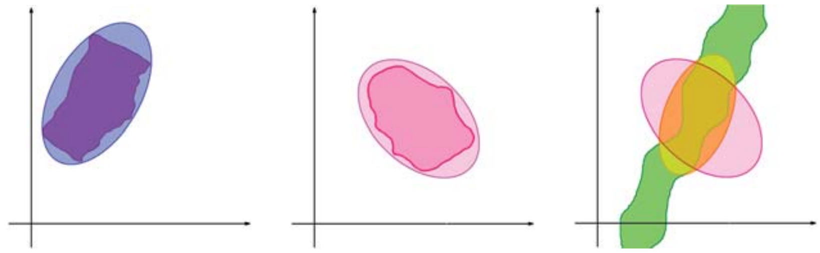
- Assumptions
  - f_k and h_k are polynomials in x_k, w_k, v_k
  - disturbances w_k and v_k are bounded by

$$\mathcal{W}_k = \{w_k \mid q_k(w_k) \geq 0\},\ q_k(w_k) \in \mathbb{R}^{n_q}$$
$$\mathcal{V}_k = \{v_k \mid r_k(v_k) \geq 0\},\ r_k(v_k) \in \mathbb{R}^{n_r},$$

  where q_k and r_k are polynomials
- Idea of Approximations
  - outer-approximate initial set and filter sets by ellipsoids

$$\mathcal{E}(\hat{x}_{i|j}, \underbrace{P_{i|j}}_{\succ 0}) = \{x \in \mathbb{R}^{n_x} \mid 1 - (x - \hat{x}_{i|j})^T P_{i|j}^{-1}(x - \hat{x}_{i|j}) \geq 0\}$$

Approximation by Ellipsoidal Sets



previous states	prediction	correction
$x_k \in \mathcal{E}(\hat{x}_{k\|k}, P_{k\|k})$	$x_{k+1} \in \mathcal{E}(\hat{x}_{k+1\|k}, P_{k+1\|k})$	$x_{k+1} \in \mathcal{E}(\hat{x}_{k+1\|k+1}, P_{k+1\|k+1})$
	for all x_{k+1} with $x_{k+1} = f_k(x_k, w_k)$, $w_k \in \mathcal{W}_k$	for all x_{k+1} with $x_{k+1} \in \mathcal{E}(\hat{x}_{k+1\|k}, P_{k+1\|k})$, $y_{k+1} = h_{k+1}(x_{k+1}, v_{k+1})$, $v_{k+1} \in \mathcal{V}_k$

Computation of the Ellipsoids

Find smallest ellipsoid $\mathcal{E}(\hat{x}_{k+1|k+1}, P_{k+1|k+1})$ with $P_{k+1|k+1} \succ 0$, that contains all x_{k+1} which are consistent with

- system dynamics $x_{k+1} = f_k(x_k, w_k)$
- previous states $x_k \in \mathcal{E}(\hat{x}_{k|k}, P_{k|k})$

$$\Leftrightarrow 1 - (x_k - \hat{x}_{k|k})^T P_{k|k}^{-1}(x_k - \hat{x}_{k|k}) := \epsilon_{k|k}(x) \geq 0$$

- disturbances $w_k \in \mathcal{W}_k$
- measurement y_{k+1}

Theorem *[Maier and Allgöwer, CDC'09]*:

The ellipsoid $\mathcal{E}(\hat{x}_{k+1|k+1}, P_{k+1|k+1})$ can be computed by solving two SOS programs.

minimize $\mathrm{tr}(P_{k+1|k})$ subject to
$P_{k+1|k} \succ 0$,
$s_1(x_k, w_k)$ is an SOS in (x_k, w_k),
$s_2(x_k, w_k)$ is an SOS in (x_k, w_k),

$$\begin{bmatrix} 1 - (s_1(x_k, w_k)\epsilon_{k|k}(x_k) + s_2^T(x_k, w_k)q_k(w_k)) & * \\ f_k(x_k, w_k) - \hat{x}_{k+1|k} & P_{k+1|k} \end{bmatrix}$$

is an SOS-matrix in (x_k, w_k)

minimize $\mathrm{tr}(P_{k+1|k+1})$ subject to
$P_{k+1|k+1} \succ 0$,
$s_3(x_{k+1}, v_{k+1})$ is an SOS in (x_{k+1}, v_{k+1}),
$s_4(x_{k+1}, v_{k+1})$ is an SOS in (x_{k+1}, v_{k+1}),
$t_5(x_{k+1}, v_{k+1})$ is a polynomial in (x_{k+1}, v_{k+1}),

$$\begin{bmatrix} 1 - (s_3(x_{k+1}, v_{k+1})\epsilon_{k+1|k}(x_k + 1) \\ \quad + s_4(x_{k+1}, v_{k+1})r_{k+1}(v_{k+1})) \\ + t_5^T(x_{k+1}, v_{k+1})(y_{k+1} - h_{k+1}(x_{k+1}, v_{k+1})) & * \\ x_{k+1} - \hat{x}_{k+1|k+1} & P_{k+1|k+1} \end{bmatrix}$$

is an SOS-matrix in (x_{k+1}, v_{k+1})

Illustrating Example: Additive Disturbance

Discrete time model of Van der Pol oscillator I

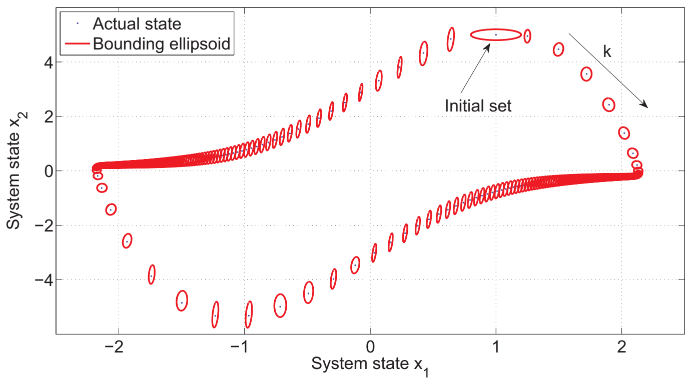
$$x_{k+1} = \begin{bmatrix} x_{1,k} + \Delta T x_{2,k} \\ x_{2,k} + \Delta T(-x_{1,k} + \mu x_{2,k}(1 - x_{1,k}^2) + w_k) \end{bmatrix}$$

$$y_k = x_{1,k} + v_k$$

$$\Delta T = 0.05, \quad \mu = 3$$

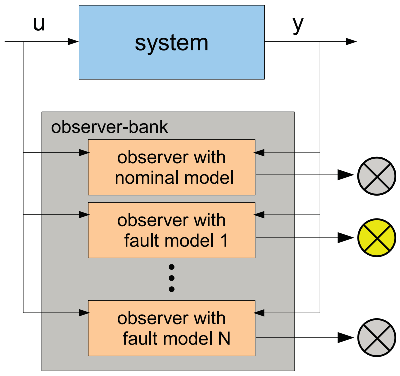
$$||w_k|| \leq 1$$
$$||v_k|| \leq 0.01$$

Illustrating Example: Additive Disturbance



Bound on ellipsoids does not grow!

Application: Fault Detection

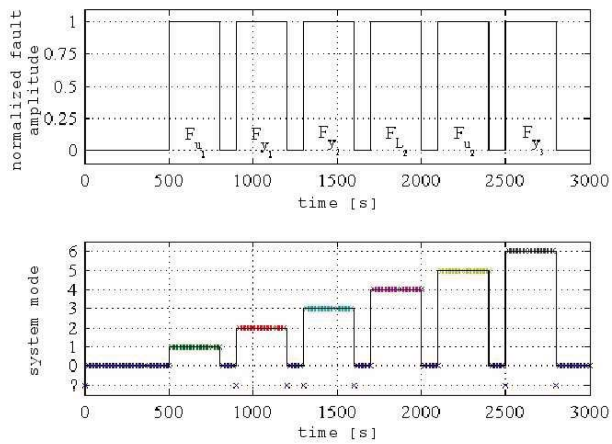


Fault Detection with Set-Based Observers

- observer yields empty set
  $\Rightarrow$ Model does not match plant
- use a whole bank of observers with models for different fault scenarios
- model that is flagged "correct" by observer bank corresponds to detected fault scenario

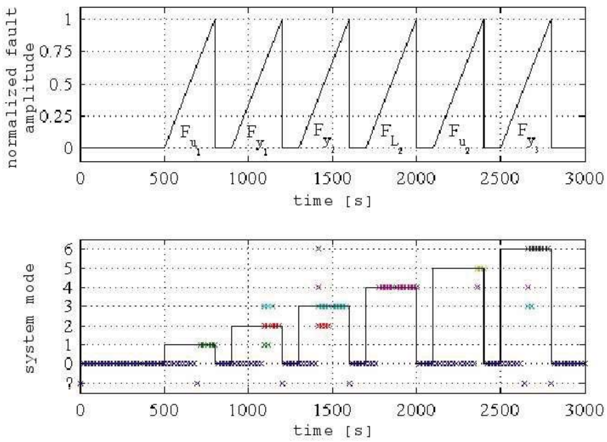
Simulation Results

- System: MIMO 3-tank system
- Fault scenarios: single faults $F_{u_1}, F_{u_2}, F_{y_1}, F_{y_2}, F_{y_3}, F_{L_2}$



Simulation Results

- System: MIMO 3-tank system
- Fault scenarios: single faults $F_{u1}, F_{u2}, F_{y1}, F_{y2}, F_{y3}, F_{L2}$



Summary and Discussion

Summary

- set-based estimation
- two-step prediction-correction procedure
- computation: ellipsoidal approach for polynomial systems based on sum of squares programming
- suited for applications in fault-detection, set-based control (tube MPC etc.), monitoring, ...

Extensions

- known inputs like e.g. control inputs
- parametric uncertainties

Summary and Discussion

Advantages Over Conventional Estimation

- allows bounds on disturbances to be taken into account
- gives bounds on actual state, not just a best guess

⇒ better adjusted to real problem settings

Disadvantages

- burdensome computations (semidefinite programs at every step)
- conservatism (ellipsoidal approximation, S-procedure, SOS-relaxation)

Conclusions

➢ **Message:**
New observer structures, that are conceptually different from the Luenberger structure, allow advanced state estimation.

➢ Exemplarily showed
  - ➢ finite time convergent observer
  - ➢ set-valued observer

  in this talk.

➢ There is comperatively little research on new estimation schemes.

➢ *Nonlinear* observers are much(!) harder to design. There are, however, very good solutions for certain classes of nonlinear systems.

➢ There are hardly any applications using alternative observer structures, despite the importance of the topic.

➢ Many open challenges: Decentralized estimation in networks, …